A Study of Certain Aspects of Generalized Special Functions, Fractional Operators and their Applications

SYNOPSIS

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CERTIFICATE

This is to certify that this Research Protocol entitled "**A Study of Certain Aspects of Generalized Special Functions, Fractional Operators and their Applications"**by **Sapna Tyagi**, submitted in fulfillment of the requirement for registration in Ph.D. Program in DEPARTMENT OF MATHEMATICS under Faculty of Science of JECRC University, Jaipur, during the academic year 2019, is a bonafide record work carried out under our guidance and supervision.

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CONTENTS

TOPICS

1. Abstract
2. Introduction
3. Literature Review
4. Description of Topic
5. Objectives of Research in bullet form
6. Research Methodology to be adopted
7. Proposed/expected outcome of the research
8. References

1. ABSTRACT

Fractional calculus is the generalization of ordinary calculus by introducing derivatives and integrals of arbitrary order, real or complex. Employing concepts of ordinary calculus and fractional calculus we can develop some new fractional /differential operators.

The special functions comprise a very old branch of mathematics. Special functions arise in the solution of partial differential equations governing the behavior of certain physical quantities. The search for a complete and unified theory of Special functions has continued since the nineteenth century. Their importance as a tool of mathematical analysis is well known to the applied scientists and engineers dealing with the practical application of differential equations, and the variety of problems leading to special functions stimulated the development of their theory.

2. INTRODUCTION

Fractional Calculus is a branch of mathematics where we learn differentiation and integration to an arbitrary order. In basic calculus we have learned the concept of differentiation and integration. For example the first order differentiation of $f(x) = x^3$ is $\frac{df(x)}{dx} = 3x^2$ and differentiating the same function to the second order result in to $\frac{d^2}{dx^2} f(x) = 6x$. Similarly $\int f(x)\,dx = \frac{1}{4}x^4 + c$ and $\int [\int f(x)\,dx]\,dx = \frac{1}{20}x^5 + c$. Now if we want to integrate our function $f(x)$ to the $\frac{1}{2}th$ order, or find its $\frac{1}{2}th$ order derivative? How could we calculate our function? Through ordinary calculus it is very typical to get a result of this function. But we may achieve the result of this function through fractional calculus.

This same question was raised by Leibniz to L`Hospital by a letter in 1695, in the following manner: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L`Hospital was more curious about that question and replied by another question to Leibniz: "What if the order will be 1/2?" On September 30, 1695 — the exact birthday of the fractional calculus! — Leibnitz in a letter replied: "It will lead to a paradox, from which one day useful consequences will be drawn." After that fractional derivative was an ongoing topic for more than 300 years. Many known mathematicians contributed to this theory over the years, among them Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov.

Beginning with, $y = x^m$, where m is positive integer .The first mathematician Lacorix found it's n^{th}derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{m-n!} x^{m-n} \text{ ,} m \geq n$$

By using Legendre's symbol Γ, for the generalized factorial , he wrote

$$\frac{d^n y}{dx^n} = \frac{\mathbb{\Gamma}(m+1)}{\mathbb{\Gamma}(m-n+1)} x^{m-n}$$

At last by letting $m = 1$ and $n = \frac{1}{2}$, got

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

In 1823, the first use of fractional operations by Abel not by the Lacorix .In the solution of integral equation, he used fractional calculus.

The theory of fractional calculus is mainly based upon the study of the familiar differential operator ${}_{\alpha}D_z^{\mu}$ defined by Miller and Ross (1993),Samko and Kilbas(1993) as:

$$ {}_{\alpha}\mathrm{D}_{\mathrm{z}}^{\mu}\left[\mathrm{f}(\mathrm{z})\right] = \frac{1}{\Gamma(-\mu)}\int_{\alpha}^{\mathrm{z}}(\mathrm{z}-1)^{-\mu-1}\,\mathrm{f}(\mathrm{t})\,\mathrm{dt}, \qquad \mathrm{Re}(\mu) < 0 \qquad \text{... (1)}$$

If Re (μ) ≥ 0, then

$$ {}_{\alpha}\mathrm{D}_{\mathrm{z}}^{\mu}\left[\mathrm{f}(\mathrm{z})\right] = \frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{dz}^{\mathrm{m}}}\left[{}_{\alpha}\mathrm{D}_{\mathrm{z}}^{\mu-\mathrm{m}}\left\{\mathrm{f}(\mathrm{z})\right\}\right], \qquad \text{... (2)}$$

where the positive integer m is such that $m - 1 \leq Re(\mu) < m$ .for $\alpha = 0$, equation (1) define the classical Rieman-Liouville fractional derivative (or intregal) of order. On the other hand, when $\alpha \to \infty$, it may be identified with te definition of the familiar Weyl fractional derivative of order.

Due to an important role played by Riemann –Liouville and Weyl integral operators in the theory of integral equation and problems of mathematical physics, a lot of generalizations of fractional integral operators are proposed from time to time by many authors.

Special functions originated from England in the seventeenth century but they were widely flourished in Germany and France in the nineteenth century. The main achievement of this country was the introduction of the Hypergeometric series given by the famous German Mathematician Gauss in the year 1812.

C. F. Gauss systematically formulated his famous infinite series as follows:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots, \qquad \ldots (3)$$

where $(a)_n = a\,(a+1)\ldots(a+n-1)$; $(a)_0 = 1$.

This series is of prime importance to mathematicians and reduces to the elementary geometric series. Hence it is called the hypergeometric series or, more precisely, Gauss's hypergeometric series and usually represented by symbol ${}_2F_1\,(a, b; c; z)$, the well-known Gauss hypergeometric function. The series in (3) is not defined if c is zero or negative integer and terminates if either a, b is zero or negative integer. The series in (3) converges for all z, real or complex, such that $|z| < 1$, diverges if $|z| > 1$, and converges absolutely for $z = 1$ if $\mathrm{Re}\,(c - a - b) > 0$, and also when $z = -1$ if $\mathrm{Re}\,(c - a - b) > -1$.

If in the equation (3), we replace z by $\frac{z}{b}$ and let $b \to \infty$, then

$$\lim_{b\to\infty} \frac{(b)_n}{b^n} = 1, \qquad \ldots (4)$$

and we arrive at the following series:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a}{c}z + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \ldots, \qquad \ldots (5)$$

The above series is represented by the symbol ${}_1F_1\,(a; c; z)$ and known as confluent hypergeometric function. The Gaussian hypergeometric function is, no doubt, of fundamental importance in the theory of special functions and mathematical physics. The importance of this function lies in the well-known fact that most of the functions that occur in the analysis are special cases of the hypergeometric functions.

Now a day's special functions are defined in several ways notably by power series, generating function, infinite product, integrals functions, the Bessel function, difference function, the Gauss Kummer's confluent and generalized hypergeometric functions, the incomplete Gamma and Beta functions and error function, Whittaker functions and so on. The theory of special functions has a long and varied history with immense literature due to their applicability as solutions of arbitrary order and the fractional order (or multi-order) differential and integral equations modeling the various phenomena arising in physical,

biological, chemical, engineering and earth sciences along with problems pertaining to automatization and economics mathematically.

3. LITERATURE REVIEW :

A major achievement in the theory of special function was the study of hyper – geometric series which was discovered by Gauss(1812) and named as Gauss hyper-geometric function . In 1946, C.S. Meijer introduced some other important generalization of special functions popularly known as Mejier's functions. A more general function known as H- function was firstly introduced by Pinchere (1888) in the form of Mellin-Barnes counter intregal .This function was further developed and studied by Fox(1961).

Generalized Bessel fuction, Wright's generalised Hypergeometric function and several others form its special function cases whereas I-functon(1982) and Aleph function (1988) are the extensions of Fox's H-function.

Fractional calculus is the generalization of ordinary calculus by introducing derivatives and integrals of arbitrary order, real or complex. Up to the middle of the 20^{th} century a lot of work had be done in the field of fractional calculus by famous mathematicians as Laplace(1812), Fourier(1822), N.H.Abel(1823-1826), J .Liouville (1832-1837), B.Riemann (1847), Grünwald(1867), M.Caputo (1967), Miller and Ross(1993),Kiryakova(1994), Podlubny(1999) etc. Notions of fractional derivatives and integrals was introduced by Agrawal(2010) and later studied by Klimek and Lupa(2013) , Odzijwicz and Bourdin etal.(2014),Ji-Huan He et.al(2016) .The well-known books by Oldham and Spanier(1974), Miller and Ross (1993), Samko, Kilbas and Marichev(1993), Kilbas and Saigo(1993), and Nishimoto(1990), on fractional calculus give a vivid account of its development and use in diverse fields of science and technology.

The applications of fractional calculus have been studied by many researchers. So name a few, the model frequency dependent damping behavior of many viscoelastic materials given by Bagley and Torvik (1983, 1986),economics given by Baillie (1996), continuum and statistical mechanics given by Mainardi (1997), bioengineering given by Magin (2004), and dynamics of interfaces between nanoparticles and rough substrates given by Chow (2005). In mechanics fractional-order derivatives have been successfully used to model damping forces with memory effect or to describe state feedback controllers is shown by Tenreiro Machado (2009).Cancer model is given by E.Ahmed (2012), Radiofrequency Ablation for Varicose Veins by Choi(2014),explanation of polar bear hairs by He (2016),Smoking dynamics given by Singh et.al (2017), Dynamical Model of Interpersonal and Romantic Relationships is given by Singh et.al (2017), A epidemiological model for computer viruses is studied by

Singh et.al (2018), The Drinfeld-Sokolov-Wilson equation is discussed by Singh et.al (2018) and a diabetes model with exponential law is analyzed by Singh et.al (2018).Thermal behavior of solid tumor given by Ghangas (2018) , Rumor spreading dynamical model in a social network given by Singh (2019) and the local fractional wave equation in fractal strings given by Singh et.al (2019).

4. DESCRIPTION OF TOPIC: -

The following advanced special functions are proposed to be studied during the course of our investigation:

THE INCOMPELETE GAMMA FUNCTION-

The incomplete gamma function is a simply generalization of a factorial for all real numbers. Its definition is given by

$$\mathbb{\Gamma}(a,x) = \int_0^x e^{-t}\, t^{a-1} dt, \quad x \epsilon R^+$$

$$\mathbb{\Gamma}(a,x) = \int_x^\infty t^{a-1}\, e^{-t} dt$$

… (6)

Like incomplete Gamma function the incomplete Beta function is defined by a definite integral as

$$B(x;a,,b) = \int_0^x t^{p-1}(1-t)^{-1} dt, \quad x, y \in R^+$$

… (7)

THE MITTAG-LEFFLER FUNCTION-

In fractional calculus it is a direct generalization of the exponential function e^x. The one and two parameter representations of the Mittag-Leffler function can be defined in terms of a power series as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\mathbb{\Gamma}(ak+\beta)} \quad \text{… (8)}$$

GENERALIZED HYPERGEMETRIC FUNCTION ${}_pF_q$

$${}_pF_q\left[a_1,\ldots, a_p\,; b_1,\ldots, b_q\,;\, z\right] = {}_pF_q\left[\begin{matrix} a_1,\ldots, a_p; \\ b_1,\ldots, b_q; \end{matrix} z\right] = {}_pF_q\left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} z\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!} \qquad \ldots (9)$$

FOX H-FUNCTION

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \theta(\xi)\, z^{\xi}\, d\xi$$

$\ldots (10)$

where $\omega = \sqrt{-1}$, $z \neq 0$ and

$$\theta(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j \xi) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)}$$

$\ldots (11)$

MULTIVARIABLE H-FUNCTION

$$H[z_1, \ldots, z_r] = H^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P,Q:P_1,Q_1;\ldots;P_r,Q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} (a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)})_{1,P} : (c_j^{(1)}, \gamma_j^{(1)})_{1,P_1}; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P_r} \\ (b_j; \beta_j^{(1)}, \ldots, \beta_j^{(r)})_{1,Q} : (d_j^{(1)}, \delta_j^{(1)})_{1,Q_1}; \ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q_r} \end{matrix} \right. \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \ldots \int_{L_r} \psi(\xi_1, \ldots, \xi_r) \prod_{i=1}^{r} \{\phi_i(\xi_i) z_i^{\xi_i}\}\, d\xi_1 \ldots d\xi_r$$

$\ldots (12)$

where

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{N_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=M_i+1}^{Q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=N_i+1}^{P_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)} ; \quad \forall i \in \{1, \ldots, r\}$$

$\ldots (13)$

and

$$\psi(\xi_1,...,\xi_r)=\frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=N+1}^{P}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)\prod_{j=1}^{Q}\Gamma(1-b_j+\sum_{j=1}^{r}\beta_j^{(i)}\xi_i)}$$

… (14)

THE I-FUNCTION

$$I_{p_i,q_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \chi(\xi)z^{\xi}d\xi$$

… (15)

where

$$\chi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}$$

… (16)

THE ALEPH-FUNCTION

$$\aleph[z]=\aleph_{p_i,q_i,\tau_i;r}^{M,N}[z]=\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left[z\left|\begin{matrix}(a_j,A_j)_{1,N},...,[\tau_i(a_j,A_j)]_{N+1,p_i}\\(b_j,B_j)_{1,M},...,[\tau_i(b_j,B_j)]_{M+1,q_i}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int_L \Phi(\xi)z^{-\xi}d\xi \qquad \text{, for all } z\neq 0, \text{ where } \omega=\sqrt{-1}$$

… (17)

and

$$\Phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-A_j\xi)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}\xi)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}\xi)}$$

… (18)

GENERAL CLASS OF POLYNOMIALS S_V^U AND ITS MULTIVARIABLE ANALOGUE $S_V^{U_1,\ldots,U_k}$

$$S_V^U[x] = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} A_{V,R}}{R!} x^R, \quad V = 0,1,2,\ldots$$

… (19)

$$S_V^{U_1,\ldots,U_k}[x_1,\ldots,x_k] = \sum_{\substack{i=1 \\ R_1,\ldots,R_k=0}}^{\substack{k \\ \sum U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V; R_1,\ldots,R_k) \frac{x_1^{R_1}}{R_1!} \ldots \frac{x_k^{R_k}}{R_k!}$$

… (20)

Fractional Calculus is a significant topic in mathematical analysis as a result of its increasing range of applications. It provides several tools for solving differential and integral equations of fractional order. The Fractional Calculus is as old as classical calculus but has gained importance during the past three decades only, due mainly to its applications in numerous diverse fields of science and engineering. A growing number of works in science and engineering deal with dynamical systems described by Fractional Order Equations that involve derivatives and integrals of non-integer order as these describe the memory and hereditary properties of different substances.

The topic Fractional calculus and its Application is a broad topic in which we study some related theory of fractional calculus and its research to use it in various physical problems.

5. OBJECTIVE OF THE STUDY :

The objectives of the proposed study are as follows:

- To study the integrals and derivatives of fractional order.
- To define new fractional order derivatives and integrals.
- To use and find the applications of fractional calculus in various fields of science and engineering.

- To study homotopy based techniques and their coupling with various integral transforms to solve the real world problems.

6. METHODOLOGY TO BE ADOPTED:

In the present work we shall solve fractional differential equations by several methods. Some of these methods use integral transform in order to reduce equations into simpler equation, and some other methods give the solutions in a series form which converges to the exact solution and some other obtain the exact and approximate analytic solutions of fractional differential equations. Some methods used are Adomian decomposition method, differential transform method and matrix method/homotopy analysis and *q*-homotopy analysis transform method.

In our work we shall use some such suitable approaches recently used by several authors in their work. We shall also use the softwares MATLAB/MAPLE/MATHEMATICA in solving equations and plotting graphs .

7. PROPOSED/EXPECTED OUTCOMES OF THE RESEARCH:

The proposed study is aimed to give following expected outcomes:

- The results from this research are expected to provide new findings to the problems involving fractional derivatives and fractional integrals.
- The study will develop suitable and well founded numerical methods to solve Fractional ordinary and partial differential equations, so that researchers can refine their results, as in the classical case.
- The work is also intended to find several other applications of fractional differential equations in both engineering and sciences such as physics, statistics, mechanics, chemistry, biology, etc.
- The research work is expected to give some life saving results in the field of medical science.

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